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STUDY OF INHERENT FREQUENCY OF HELMHOLTZ RESONATOR

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ABSTRACT

Self-resonating water jet has been the subject of interest for researchers. The inherent frequency of Helmholtz Resonator is one of the important parameters for design of self-resonating water jet nozzle device. A new parametric model for prediction of the inherent frequency of Helmholtz resonator used in the field of water jet technology was proposed in this paper. Development of the model was based on the assumptions that the length and the diameter of Helmholtz resonating chamber and the length of the straight pipe segment of water jet nozzle are in the same order of magnitude. In comparison with the existing parametric model, the assumptions are more reasonable and the physical model on which the new parametric model was developed is in good agreement with the real configuration of Helmholtz resonator found in self-resonating water jet. As a result, the new model was expected to be more accurate in prediction of the inherent frequency of Helmholtz resonator.

1. INTRODUCTION

Compared with continuous high pressure water jet, pulsed high water jet has higher jetting efficiency. The principle of self-resonating was widely used for generating the pulsed water jet. The inherent frequency of the Helmholtz resonator is the key parameter for the self-resonating pulsed water jet technology. Fluid self-resonance occurs under the condition that the induction frequency (pressure agitation frequency) is consistent with the inherent frequency of Helmholtz resonator. Therefore, determination of the inherent frequency of Helmholtz resonator is the interested subject in this field.

The widely used mathematical model for prediction of the natural frequency of Helmholtz resonator was given by [1]:

$$f = \frac{c_*}{2\pi D_2}\sqrt{\frac{D_1}{L_1}} \tag{1.1}$$

Where f is the inherent frequency of Helmholtz resonator, A_1 is the water nozzle diameter, L_1 is the length of inlet nozzle's straight pipe segment, D_1 is the diameter of the inlet nozzle, D_2 is the diameter of the resonating chamber, and c* is the overall velocity of agitation wave in elastic tube. The definition of c* is given by:

$$c_*^2 = \frac{1}{\frac{1}{a^2} + \frac{1}{c_0^2}} \tag{1.2}$$

Where a is the wave velocity of in water and c_0 denote the wave velocity of in elastic tube wall.

The physical model from which Eq. (1.1) was developed is shown in Fig. 1. This model development was based on the assumption that self-resonating chamber is a concentrated

Parameter, the length of the chamber is much less than its diameter and can therefore be neglected. This assumption, however, does not agree well with the typical structure of self-resonator used in self-resonating water jet technologies, consequently the model is expected to have some deviation in predictions. Therefore, development of a new prediction model with higher prediction accuracy is required and provides the motivation for the study presented in this paper.

2. SOLUTION OF GOVERNING EQUATIONS OF TRANSIENT FLOW

2.1 Governing equations of transient flow in tube

The segment of a round-sectioned tube with A being its sectional area was chosen as control volume (see Fig. 2). The length of control volume is Δx, the abscissa of inlet plane is x, the abscissa of outlet plane is x+Δx. It is apparent that x and Δx are independent of time t, ρ is the density of water, and u is the axial velocity of the fluid.

In the case of flow in the tube with round cross section, $u_y=u_z=0$, $u_x=u=u(x)$, the cross sectional area is a function of x and t, i. e., A=A(x,t), owing to the elasticity of the tube wall. The continuity equation for the elastic tube can therefore be written as:

$$\frac{\partial \rho A}{\partial t}+\frac{\partial \rho A u}{\partial x}=0 \tag{2.1}$$

It was assumed that flow was continuous, incompressible, isothermal, frictionless, one dimensional, without body forces, and flowing within the tube of round cross section, Navier-Stokes equation can be written as:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.2}$$

In the case of transient flow and fluid resonance, inertia force becomes the major factor and the viscosity of the fluid can be neglected. Therefore, Eq. (2.1) and Eq. (2.2) are the governing equations for the study of transient flow in the tube.

2.2 Linearization of the governing equations

The velocity of the wave in water represents the influence of water compressibility upon the propagating velocity of the agitation wave, which is defined as:

$$a^2=\frac{dp}{d\rho} \tag{2.3}$$

The velocity of the wave in elastic tube wall resents the influence of the elasticity of tube wall upon the propagating velocity of the agitation wave, which is defined as:

$$c_0^2 = \frac{A}{\rho}\frac{dp}{dA} \tag{2.4}$$

According Eq. (2.3), Eq. (2.4) and Eq. (1.2), the continuity equation (2.1) can be transformed into:

$$\frac{A}{c_*^2}\frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{2.5}$$

Where q=ρAu is the mass flow rate. Let q_c denote integral of the first term of Eq. (2.5) along the tube segment L. When L is small, the first term can be considered to be constant. Then one obtains:

$$q_c = \int_0^L \frac{A}{c_*^2}\frac{\partial p}{\partial t}dx = \frac{A}{c_*^2}\frac{\partial p}{\partial t}L \tag{2.6}$$

The fluid capacitance per unit tube length is defined as:

$$C = \frac{q_c}{\frac{\partial p}{\partial t}} = \frac{A}{c_*^2} \tag{2.7}$$

The final form of the linearized continuity equation is given by:

$$C\frac{\partial p}{\partial t} = -\frac{\partial q}{\partial x}. \tag{2.8}$$

Multiply the continuity equation (2.1) by u, and the Navier-Stokes equation (2.2) by ρA, then added together, Eq. (2.2) becomes:

$$\frac{\partial q}{\partial t} + \frac{\partial qu}{\partial x} = -A\frac{\partial p}{\partial x} \tag{2.9}$$

Assuming that the flow velocity of the fluid is much less than the propagating velocity of the agitation wave, the second term of migration acceleration of ∂qu/∂x in Eq. (2.9) can be neglected. Hence the Navier-Stokes equation (2.9) is linearized as:

$$l\frac{\partial q}{\partial t} = -\frac{\partial p}{\partial x} \tag{2.10}$$

Where l is the fluid inductance defined by:

$$l = \frac{\Delta p}{\frac{\partial q}{\partial t}} = \frac{1}{A} \tag{2.11}$$

Where Δp is the pressure drop along the tube segment L. The ideal fluid one-dimensional linearization governing equations for transient flow in the tube are achieved by a sequence of mathematical manipulations, as given by Eq. (2.8) and Eq. (2.10).

2.3 Solution of governing equations with fluid impedance method

According to the assumptions, the pressure p can be expressed as: p=p(x,t) and mass flow rate as: q=q(x,t). The corresponding Laplace transformations can be given as: P(x,s)=L[p(x,t)], and Q(x,s)=L[q(x,t)]. The initial conditions are:

$$q(x,0) = 0 \qquad\qquad p(x,0) = 0 \tag{2.12}$$

Alternating Eq. (2.8) and Eq. (2.10) by Laplace transformation yields:

$$\begin{cases} \dfrac{\partial P(x,s)}{\partial x} + lsQ(x,s) = 0 \\ \dfrac{\partial Q(x,s)}{\partial x} + CsP(x,s) = 0 \end{cases} \tag{2.13}$$

Eq. (2.13) is the linearized and Laplace-transformed governing equations.

The fluid resistance is defined as: R=Δp/q.   Its Laplace transformation of Z=P/Q is noted as the fluid impedance. The fluid capacitance C representing fluid compressibility and tube wall's elasticity, as well as the fluid inductance l caused by the unsteady flow, will exhibit reactance, as the fluid flowing through the pipe system. Laplace transformation of expression (2.7) (fluid capacitance) leads to C=Qc/sP. Similarly, the capacitive reactance is defined as Z*c=P/Qc=1/Sc. And by Laplace transformation of Eq. (2.11) we obtain l=P/sQ. Thus the definition of inductive reactance is: Z_l=P/Q=ls.

Under the boundary conditions of x=0, P_1=P(0,s), Q_1=Q(0,s): solution to the governing equation (2.13) is:

$$\begin{cases} P(s,x) = P_1 ch\dfrac{sx}{c_*} - Z_c Q_1 sh\dfrac{sx}{c_*} \\ Q(x,s) = -\dfrac{P_1}{Z_c} sh\dfrac{sx}{c_*} + Q_1 ch\dfrac{sx}{c_*} \end{cases} \tag{2.14}$$

Where Zc=c*/A is the characteristic impedance of the tube. At the end of the tube (x=L), the pressure and flow rate are:

$$P_2 = P(s,L), Q_2 = Q(s,L) \tag{2.15}$$

Substituting Eq. (2.15) into Eq. (2.14), and letting s=jω, by algebraic manipulations we obtain:

$$\begin{cases} P_1 = P_2 \cos\dfrac{\omega L}{c_*} + Q_2 Z_c j\sin\dfrac{\omega L}{c_*} \\ Q_1 = \dfrac{P_2}{Z_c} j\sin\dfrac{\omega L}{c_*} + Q_2 \cos\dfrac{\omega L}{c_*} \end{cases} \tag{2.16}$$

And further we can obtain the solution to the governing equation for one dimensional transient flow in the tube by fluid impedance method[2]:

$$Z_1 = \frac{Z_2 + Z_c jtg\dfrac{\omega L}{c_*}}{-Z_2 tg\dfrac{\omega L}{c_*} + Z_c} Z_c \tag{2.17}$$

Where Z_1=P_1/Q, Z_2=P_2/Q, are the fluid impedance at the beginning and the end of the tube respectively, ω is the circular frequency.

2.4 The inherent frequency of Helmholtz resonator

The typical structure of Helmholtz resonator found in self-resonating water jet technologies indicates that the length and diameter of resonating chamber and length of the straight pipe segment of the jet nozzle are same order of magnitude. The physical model of Helmholtz resonator is shown in Fig. 3. The straight pipe segment stands for the water jet nozzle's straight segment.

The length of straight segment is L_1, the diameter D_1, the characteristic impedance Zc_1. Z_1 and Z_2 are the fluid impedance at the beginning and the end of straight segment respectively. The length of Helmholtz resonator is L_2, the diameter D_2, the characteristic impedance Zc_2. Z_3 and Z_4 are the fluid impedance at the beginning and the end of the chamber respectively.[3]

Applying Eq. (2.17) to the straight pipe segment, then we obtain:

$$Z_1 = \frac{Z_2 + jZ_{c1} tg \frac{\omega L_1}{c_*}}{jZ_2 tg \frac{\omega L_1}{c_*} + Z_{c1}} Z_{c1} \tag{2.18}$$

Applying Eq. (2.17) to the Helmholtz chamber in the same way, we obtain:

$$Z_3 = \frac{Z_4 + jZ_{c2} tg \frac{\omega L_2}{c_*}}{jZ_4 tg \frac{\omega L_2}{c_*} + Z_{c2}} Z_{c2} \tag{2.19}$$

The conditions under which resonance takes place in the pipe system can be described by: $Z_1=0$ and $Z_4= \infty$, which was substituted into Eq. (2.18) and Eq. (2.19) respectively, and then we obtain:

$$Z_2 = -jZ_{c1} tg \frac{\omega L_1}{c_*} \tag{2.20}$$

$$Z_3 = \frac{Z_{c2}}{jtg \frac{\omega L_2}{c_*}} \tag{2.21}$$

Where $\omega=2\pi f$, f is the inherent frequency of Helmholtz resonator. Since Eq. (2.20) and Eq. (2.21) are related by: $Z_2=Z_3$, the following relation was achievable:

$$tg\frac{2\pi fL_1}{c_*}tg\frac{2\pi fL_2}{c_*}=\left(\frac{D_1}{D_2}\right)^2 \tag{2.22}$$

Where $Zc_1=c^*/A_1$ and $Zc_2=c^*/A_2$ are the characteristic impedance of the straight pipe segment and Helmholtz resonating chamber, and the corresponding cross sectional area are $A_1=\pi D_1^2/4$ and $A_2=\pi D_2^2/4$. Since L_1 and L_2 are same order of magnitude, and much less than the overall velocity of agitation wave c*, Eq. (2.22) can be simplified into:

$$tg\frac{2\pi fL_1}{c_*}tg\frac{2\pi fL_2}{c_*}=\left(\frac{2\pi f}{c_*}\right)^2 L_1L_2 \tag{2.23}$$

Substituting Eq. (2.23) into Eq. (2.22), the parametric model of the inherent frequency of Helmholtz resonator is obtained as:

$$f=\frac{c_*}{2\pi\sqrt{\lambda}D_2}\sqrt{\frac{D_1}{L_2}} \tag{2.24}$$

Where the penalty parameter λ is the ratio of the straight pipe segment to the diameter of the water jet nozzle, i.e. $\lambda=L_1/D_1$. For self-resonating water jet, Strouhal number is given by:

$$S_d=\frac{fD_1}{u} \tag{2.25}$$

Where f is the dominant frequency of the separated vortices, u is the water jetting velocity. Mach number is defined as $M=u/c^*$. Combining Eq. (2.24) and Eq. (2.25) and eliminating f, we can therefore obtain the relationship between the structural parameters and the Strouhal number and Mach number, given by:

$$\frac{D_2}{D_1}=\frac{1}{2\pi\sqrt{\lambda}S_dM}\sqrt{\frac{D_1}{L_2}} \tag{2.26}$$

3. CONCLUSIONS

A new parametric model for predicting the natural frequency of Helmholtz resonator was developed, and the relationship between the structural parameters and Strouhal number and Mach number was obtained. The assumptions on which the model development based agree with Helmholtz resonator used in self-resonating water jet, and is expected to be more accurate in predictions. The obtained results would be of help to understanding the mechanism of self-resonating water jet.

4. ACKNOWLEDGMENTS

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5. REFERENCES

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6. NOMENCLATURE

f -Inherent frequency of Helmholtz resonator
A_1-Water nozzle diameter
L_1-Llength of inlet nozzle’s straight pipe segment
D_1-Diameter of the inlet nozzle
D_2-Diameter of the resonator chamber
c*-Overall velocity of agitation wave in elastic tube
a-the wave velocity of in water
c_0-the wave velocity of in elastic tube wall
ρ-Density of water
u-Axial velocity of the fluid.
q-Mass flow rate

C-Fluid capacitance per unit tube length
l-Fluid inductance
R-Fluid resistance
Z-Fluid impedance.
Zc-Characteristic impedance of the tube
ω-Circular frequency.
λ-Ratio of the straight pipe segment to the diameter of the water jet nozzle
S_d- Strouhal number
M-Mach number

7. GRAPHICS

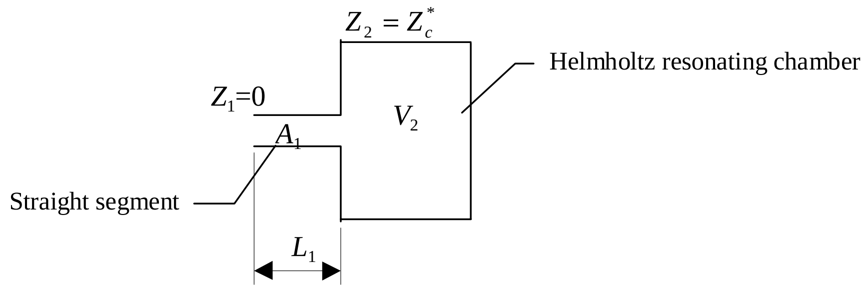

Fig. 1 The existing physical model of Helmholtz resonator

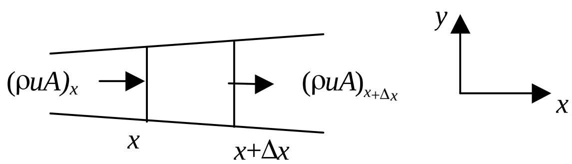

Fig. 2. Control volume in the tube with round cross section

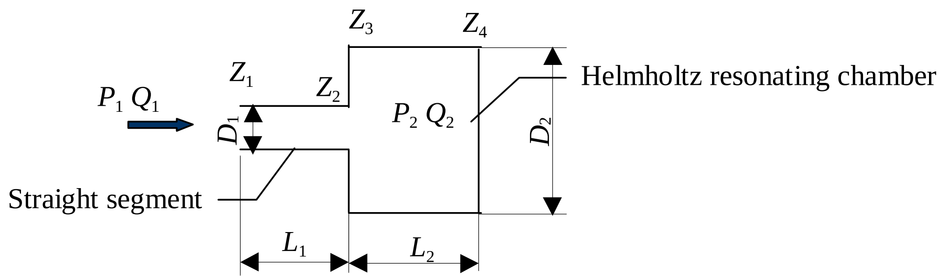

Fig. 3 Physical model of Helmholtz resonator